Exact closed form of the return probability on the Bethe lattice

## LETTER TO THE EDITOR

# Exact closed form of the return probability on the Bethe lattice

Achille Giacometti

Institut für Festkörperforschung der Kernforschungsanlange, Postfach 1913, D-52425, Jülich, Germany

**Abstract.** An exact closed-form solution for the return probability of a random walk on the Bethe lattice is given. The long-time asymptotic form confirms a previously known expression. It is, however, shown that this exact result reduces to the proper expression when the Bethe lattice degenerates on a line, unlike the asymptotic result which is singular. This is shown to be an artefact of the asymptotic expansion. The density of states is also calculated.

Besides being an interesting type of graph *per se*, the Bethe lattice (BL) is also reckoned to be a paradigm of a lattice in the limit of high dimensionality. A BL (see figure 1) is usually defined as a set of sites connected by bonds such that each site has the same coordination number and there are no closed loops. It differs from the so-called Cayley tree by the fact that the complication arising by the boundary conditions is neglected [1].
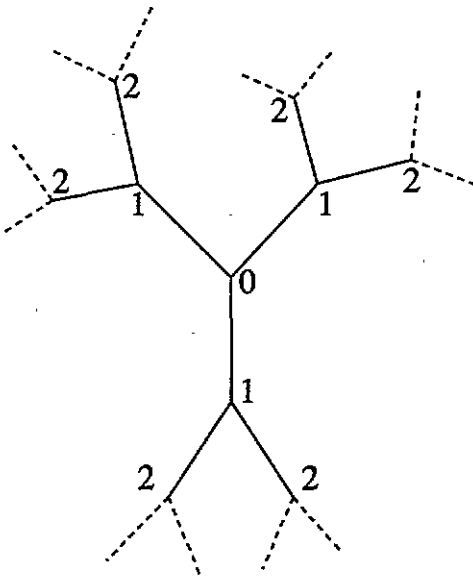


**Figure 1.** Example of a BL with coordination number $z = 3$. Sites indicated with the same numbers $n = 0, 1, 2, \ldots$ belong to the same shell.

The problem of the random walk on a BL is not new [2,3]. However, up to now only asymptotic expressions were given. One of the surprising features of these asymptotic

expressions was the difficulty arising in the interpretation of the result in the limit when the BL collapses into a line.

The main aim of this letter is to derive an exact closed form solution which, on one hand, confirms previous asymptotic results, but, on the other hand, has the proper limit form when the BL reduces to a one-dimensional lattice, thus confirming the exactness of the asymptotic procedure and solving the aforementioned interpretation puzzle. Moreover, we will provide an alternative-solution approach with respect to the previous investigations.

Let us start from the general master equation on the lattice:

$$P_{x,0}(t+1) = P_{x,0}(t) + \sum_{y(x)}[w_{x,y}P_{y,0}(t) - w_{y,x}P_{x,0}(t)] \tag{1}$$

where $P_{x,0}(t)$ is the probability density of being at site $x$ at time $t$ having started from site 0 at the initial time $t_0 = 0$. The notation $y(x)$ means that the sum is restricted to the nearest neighbours $y$ of $x$. In the BL case, $w_{x,y} = 1/z$, where $z$ is the coordination number of the lattice.

It is convenient to introduce the generating function of $P_{x,0}(t)$ (Green function):

$$\tilde{P}_{x,0}(\lambda) = \sum_{t=0}^{+\infty} \lambda^t P_{x,0}(t).$$

The fact that all points belonging to the same shell are topologically equivalent allows us to map the solution for the BL onto the solution of a one-dimensional lattice with a defect. Therefore, the Green equation takes, on the BL, the form

$$\tilde{P}_{0,0}(\lambda) = \lambda \tilde{P}_{1,0}(\lambda) + 1 \tag{3a}$$

$$\tilde{P}_{n,0}(\lambda) = \frac{\lambda}{z}\tilde{P}_{n-1,0}(\lambda) + \frac{\lambda(z-1)}{z}\tilde{P}_{n+1,0}(\lambda) \tag{3b}$$

for the zeroth and $n$th shell, respectively. Here, $P_{n,0}(t)$ refers to the probability of being in the $n$th shell at time $t$ having started from seed 0.

The solution of equations (3a) and (3b) is considerably simplified by noting that the ratio $\tilde{P}_{n+1,0}(\lambda)/\tilde{P}_{n,0}(\lambda)$ is *independent* of $n$ due to the homogeneity of the lattice and the particular boundary conditions†. It is then a simple matter to solve the quadratic equation resulting from (3b) and substitute the root, which has finite value in the $\lambda \to 0$ limit, into (3a) with the result

$$\tilde{P}_{0,0}(\lambda) = \frac{2(z-1)/z}{(z-2)/z + \sqrt{1 - (4\lambda^2(z-1)/z^2)}}. \tag{4}$$

This result was previously obtained by Cassi [3] by a different procedure.

It is important to notice that, for $z = 2$, this expression reduces to the well known result of the generating function for the one-dimensional lattice [5]. It is also worth mentioning that, since the critical value for the fugacity $\lambda_c = z/2\sqrt{z-1} > 1$ for $z \geqslant 3$, the generating function $\tilde{P}_{0,0}(\lambda)$ is always real and finite for $\lambda \leqslant 1$ and, therefore, a random walk on the Bethe lattice *cannot* be critical.

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† This observation appears in [4] in the context of the Anderson localization.

Upon series expansion of equation (4) and using definition (2), one obtains after some algebra

$$P_{0,0}(2t) = \frac{(z-1)}{z}\left(\frac{\sqrt{z-1}}{z}\right)^{2t}\sum_{p=0}^{+\infty}\frac{(2p+2t)!}{(p+t+1)!(p+t)!}\left(\frac{z-1}{z^2}\right)^p \qquad (5)$$

and $P_{0,0}(2t+1) = 0$ for $t > 0$ with $P_{0,0}(0) = 1$. After some manipulations this expression can be cast in the following closed form:

$$P_{0,0}(2t) = \frac{(z-1)}{z}\left(\frac{\sqrt{z-1}}{z}\right)^{2t}\frac{\Gamma(2t+1)}{\Gamma(t+2)\Gamma(t+1)}\,{}_2F_1(t+\tfrac{1}{2},1,t+2,4(z-1)/z^2) \qquad (6)$$

where $\Gamma(t)$ is the gamma function and ${}_2F_1(\alpha,\beta,\gamma,z)$ is the Gauss hypergeometric function [6].

For large $t$, one can use the property†

$$_2F_1(\alpha,\beta,\beta,z) = (1-z)^{-\alpha} \qquad (7)$$

valid for arbitrary $\beta$ and the Stirling approximation for the gamma function [6] to find, at the leading order in $t \gg 1$,

$$P_{0,0}(t) \overset{t\gg 1}{\approx} \frac{2^{3/2}z(z-1)}{\sqrt{\pi}(z-2)^2}t^{-3/2}\exp\left(-t\ln\left(\frac{z}{2\sqrt{z-1}}\right)\right) \qquad (8)$$

which confirms the asymptotic result derived in [3]. Note that in the limit $z \to 2$ (when the BL degenerates on a line), the expected asymptotic behaviour $P_{0,0}(t) \sim t^{-1/2}$ is *not* recovered, both because the prefactor becomes singular and because the (universal) power law is not correct.

We are now in the position to show that this is an artefact of the asymptotic expansion stemming from the fact that the limit $t \to \infty$ and $z \to 2$ do not commute. Indeed, if we set $z = 2$ in the exact result (6) and use the fact that [6]

$$_2F_1(\alpha,\beta,\gamma,1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \qquad (9)$$

we obtain

$$P_{0,0}(t) = (1/2)^t\frac{\Gamma(t+1)}{\Gamma(t/2+1)\Gamma(t/2+1)} \qquad (10)$$

which is the well known result for the one-dimensional case [5]. This latter result could in fact be derived by starting from equation (4) for $z = 2$ and using a procedure similar to the one employed here.

It is also possible to compute the density of states, associated with master equation (1), on the BL. This was not previously calculated. Indeed, using the transformation procedure between discrete and continuum times, described in [7], it is not hard to see that the Laplace transform of the return probability is related to the generating function $\tilde{P}_{0,0}(\lambda)$ by

$$\tilde{P}_{0,0}(\omega) = \lambda\tilde{P}_{0,0}(\lambda)|_{\lambda=1/(1+\omega)} \qquad (11)$$

† A more rigorous procedure involving a Kummer transformation yields the same result.
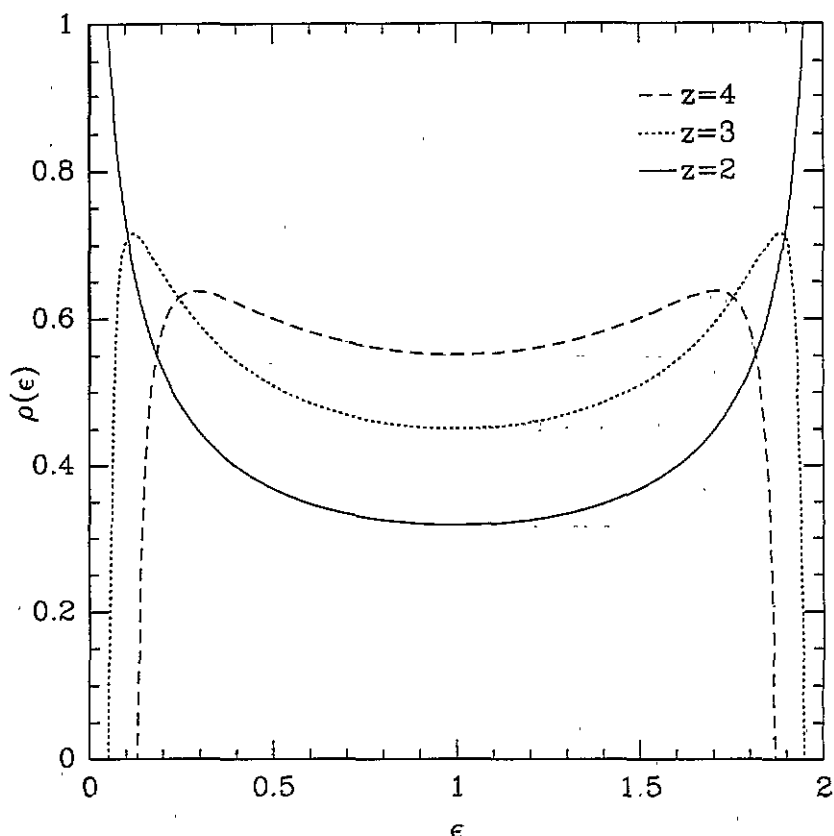
**Figure 2.** Density of states associated to the master equation in the case of a one-dimensional lattice ($z = 2$) and of a Bethe lattice ($z = 3, 4$).

where $\omega$ is the Laplace variable conjugate of time. Then, using equations (4) and (11), one obtains

$$\tilde{P}_{0,0}(\omega) = \frac{2(z-1)/z}{((z-2)/z)(1+\omega) + \sqrt{\omega^2 + 2\omega + (z-2)^2/z^2}}. \tag{12}$$

The density of state $\rho(\epsilon)$ is then well known [8] to be given by the analytical continuation

$$\rho(\epsilon) = -\frac{1}{\pi} \operatorname{Im} \tilde{P}_{0,0}(-\epsilon + i0^+). \tag{13}$$

Then, in the present case, the result of the analytical continuation is

$$\rho(\epsilon) = \begin{cases} \dfrac{z}{2\pi} \dfrac{\sqrt{2\epsilon - \epsilon^2 - (z-2)^2/z^2}}{2\epsilon - \epsilon^2} & \text{if } 2\epsilon - \epsilon^2 - (z-2)^2/z^2 > 0 \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

which, for $z = 2$, reduces to the well known formula for the density of states of a one-dimensional lattice [9]. Figure 2 shows the comparison between the case of the one-dimensional lattice ($z = 2$) and the BL ($z = 3, 4$).

In summary, we have presented an exact closed-form solution for the return probability and the density of states on the BL. Previous asymptotic results were confirmed and explained in terms of this solution. Although other relevant quantities, beside the ones presented here, could, in principle, be obtained, the required algebra rapidly becomes very involved. This is beyond the purposes of the present work; the main objective of which is the confirmation of the asymptotic results, along with the removal of the inconsistency contained in them. As a by-product of our investigation, we have also presented an alternative simplified solution procedure with respect to the previous two approaches.

## References

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[9]   see, e.g., Alexander S, Bernasconi J, Schneider W R and Orbach R 1981 *Rev. Mod. Phys.* **53** 175